

COMPRESSIBLE NAVIER-STOKES SYSTEM : LARGE SOLUTIONS AND INCOMPRESSIBLE LIMIT

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ABSTRACT. Here we prove the existence of global in time regular solutions to the two-dimensional compressible Navier-Stokes equations supplemented with arbitrary large initial velocity v_0 and almost constant density ϱ_0 , for large volume (bulk) viscosity. The result is generalized to the higher dimensional case under the additional assumption that the strong solution of the classical incompressible Navier-Stokes equations supplemented with the divergence-free projection of v_0 , is global. The systems are examined in \mathbb{R}^d with $d \geq 2$, in the critical $\dot{B}_{2,1}^s$ Besov spaces framework.

1. INTRODUCTION

We are concerned with the following compressible Navier-Stokes equations in the whole space \mathbb{R}^d :

$$(1.1) \quad \begin{cases} \varrho_t + \operatorname{div}(\varrho v) = 0, \\ \varrho v_t + \varrho v \cdot \nabla v - \mu \Delta v - (\lambda + \mu) \nabla \operatorname{div} v + \nabla P = 0, \end{cases}$$

supplemented with initial data: $\varrho|_{t=0} = \varrho_0$ and $v|_{t=0} = v_0$.

The pressure function P is given and assumed to be strictly increasing. The shear and volume viscosity coefficients λ and μ are constant (just for simplicity) and fulfill the standard strong parabolicity assumption:

$$(1.2) \quad \mu > 0 \quad \text{and} \quad \nu := \lambda + 2\mu > 0.$$

Starting with the pioneering work by Matsumura and Nishida [18, 19] in the beginning of the eighties, a number of papers have been dedicated to the challenging issue of proving the global existence of strong solutions for (1.1) in different contexts (whole space or domains, dimension $d = 2$ or $d \geq 3$, and so on). One may mention in particular the works by Zajączkowski [26], Shibata [13], Danchin [4], Mucha [21, 23, 24] and, more recently, by Kotschote [14, 15]. The common point between all those papers is that the initial velocity is assumed to be small, and that the initial density is close to a stable constant steady state.

Our main goal is to prove the global existence of strong solutions to (1.1) for a class of large initial data. In the two-dimensional case, we establish that, indeed, for fixed shear viscosity μ and any initial velocity-field v_0 (with critical regularity), the solution to (1.1) is global if λ is sufficiently large, and ϱ_0 sufficiently close (in terms of λ) to some positive constant (say 1 for notational simplicity). This result will strongly rely on the fact that, at least formally, the limit velocity for $\lambda \rightarrow +\infty$ satisfies the incompressible Navier-Stokes equations:

$$(1.3) \quad \begin{cases} V_t + V \cdot \nabla V - \mu \Delta V + \nabla \Pi = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} V = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ V|_{t=0} = V_0 & \text{at } \mathbb{R}^d, \end{cases}$$

with V_0 being the Leray-Helmholtz projection of v_0 on divergence-free vector-fields.

We are also interested in similar results in dimension $d \geq 3$. However, as in that case the global existence issue of strong solutions for (1.3) supplemented with general data is open, we have to assume first that V_0 generates a global strong solution to (1.3), and then to analyze the stability of that solution in the setting of the compressible model (1.1) with large λ .

Let us first consider the two-dimensional case, assuming that initial data ϱ_0 and v_0 fulfill the critical regularity assumptions of [4], namely¹

$$a_0 := (\varrho_0 - 1) \in \dot{B}_{2,1}^0 \cap \dot{B}_{2,1}^1(\mathbb{R}^2) \quad \text{and} \quad v_0 \in \dot{B}_{2,1}^0(\mathbb{R}^2).$$

Then the initial data V_0 of (1.3) is in $\dot{B}_{2,1}^0(\mathbb{R}^2)$. Therefore, in light of the well-known embedding $\dot{B}_{2,1}^0(\mathbb{R}^2) \subset L_2(\mathbb{R}^2)$, we are guaranteed that it generates a unique global solution V in the energy class

$$V^{1,0}(\mathbb{R}^2 \times \mathbb{R}_+) := \mathcal{C}_b(\mathbb{R}_+; L_2(\mathbb{R}^2)) \cap L_2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^2)),$$

that satisfies the energy identity:

$$\|V(t)\|_{L_2}^2 + 2\mu \int_0^t \|\nabla V\|_{L_2}^2 d\tau = \|V_0\|_{L_2}^2.$$

Based on that fact, one may prove that the additional regularity of V_0 is preserved through the time evolution (see Theorem 4.1 in the Appendix), that is

$$(1.4) \quad V \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap L_1(\mathbb{R}_+; \dot{B}_{2,1}^2(\mathbb{R}^2)).$$

Let us now state our main existence result for (1.1) in the two-dimensional setting.

Theorem 1.1. *Let $\mu \leq \nu$. Let $v_0 \in \dot{B}_{2,1}^0(\mathbb{R}^2)$ and ϱ_0 such that $a_0 := (\varrho_0 - 1) \in \dot{B}_{2,1}^0 \cap \dot{B}_{2,1}^1(\mathbb{R}^2)$. There exists a large constant C such that for $V_0 = \mathcal{P}v_0$, the divergence-free part of the initial velocity, we set*

$$(1.5) \quad M := C \|V_0\|_{\dot{B}_{2,1}^0} \exp\left(\frac{C}{\mu^4} \|V_0\|_{L_2}^4\right)$$

and if ν satisfies

$$Ce^{CM} (\|a_0\|_{\dot{B}_{2,1}^0} + \nu \|a_0\|_{\dot{B}_{2,1}^1} + \|\mathcal{Q}u_0\|_{\dot{B}_{2,1}^0} + M^2 + \mu^2) \leq \sqrt{\nu} \sqrt{\mu},$$

where \mathcal{Q} stands for the projection operator on potential vector-fields, then there exists a unique global in time regular solution (ϱ, v) to (1.1) such that

$$(1.6) \quad \begin{aligned} v &\in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^0), \quad v_t, \nabla^2 v \in L_1(\mathbb{R}_+; \dot{B}_{2,1}^0), \\ a &:= (\varrho - 1) \in \mathcal{C}(\mathbb{R}_+; \dot{B}_{2,1}^0 \cap \dot{B}_{2,1}^1) \cap L_2(\mathbb{R}_+; \dot{B}_{2,1}^1). \end{aligned}$$

In addition, the following bound is fulfilled by the solution:

$$\begin{aligned} &\|\mathcal{Q}v\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^0)} + \|\mathcal{Q}v_t, \nu \nabla^2 \mathcal{Q}v\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^0)} + \|a\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^0)} + \nu \|a\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^1)} \\ &+ \nu^{1/2} (\|\mathcal{P}v - V\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^0)} + \|\mathcal{P}v_t - V_t, \mu \nabla^2 (\mathcal{P}v - V)\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^0)}) \\ &\leq Ce^{CM} (\|a_0\|_{\dot{B}_{2,1}^0} + \nu \|a_0\|_{\dot{B}_{2,1}^1} + \|\mathcal{Q}u_0\|_{\dot{B}_{2,1}^0} + M^2 + \mu^2). \end{aligned}$$

¹The reader may refer to the next section for the definition of homogeneous Besov spaces $\dot{B}_{2,1}^s(\mathbb{R}^d)$.

Let us emphasize that in contrast with the global existence results cited above, we do not require any smallness condition on the initial velocity : the volume viscosity λ just has to be sufficiently large. The mechanism underneath is that having large λ provides strong dissipation on the potential part of the velocity, and thus makes our flow almost incompressible. At the same time, strong dissipation does not involve the divergence free part of the flow, but as we are in dimension two, it is known that it generates a global strong incompressible solution. In fact, our statement may be seen as a stability result for incompressible flows within compressible flows.

Our result has some similarity with that of the first author in [5, 6] where it is shown that large initial velocities give rise to global strong solutions in the low Mach number asymptotics. However, the mechanism leading to global existence therein strongly relies on the dispersive (or highly oscillating) properties of the acoustic wave equations. This is in sharp contrast with the situation we are looking at here, where dispersion completely disappears when $\lambda \rightarrow +\infty$.

There are also examples of large data generating global strong solutions to the compressible Navier-Stokes equations, independently of any asymptotic considerations. In this regard, one has to mention the result by Kazhikov-Weigant [12] in the two dimensional case, where it is assumed that the volume viscosity λ has some suitable dependence with respect to the density the density (like $\lambda(\varrho) = \varrho^\beta$ for some $\beta > 3$). In contrast, here we do not require any particular nonlinear structure of the viscosity coefficients, but rather that the volume viscosity is large enough. Finally, in a recent paper [11] dedicated to the shallow water equations (that is μ depends linearly on ϱ and $\lambda = 0$), B. Haspot established the existence of global strong solutions allowing for large potential part of the initial velocity.

As a by product of Theorem 1.1, we get that $(\varrho, v) \rightarrow (1, V)$ with a convergence rate of order $\nu^{-1/2}$. This is stated more exactly in the following corollary.

Corollary 1.1. *Let v_0 be any vector field in $\dot{B}_{2,1}^0(\mathbb{R}^2)$, and M be defined by (1.5). Then for large enough ν (or equivalently λ), System (1.1) supplemented with initial density 1 and initial velocity v_0 has a unique global solution (ϱ, v) in the space given by (1.6). Furthermore, if V stands for the solution to (1.3) then $(\varrho, v) \rightarrow (1, V)$ as follows:*

$$\begin{aligned} \|\varrho - 1\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^1)} + \|\nabla^2 \mathcal{Q}v\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^0)} + \|\mathcal{P}v - V\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^0)} \\ + \|\mathcal{P}v_t - V_t, \mu \nabla^2(\mathcal{P}v - V)\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^0)} \leq C\nu^{-1/2} \sqrt{\mu}. \end{aligned}$$

Let us now describe our main result in the high-dimensional case $d \geq 3$. Then it turns out that our approach for exhibiting large global solutions is essentially the same, once it is known that the limit system (1.3) supplemented with initial data $V_0 := \mathcal{P}v_0$ has a global strong solution with suitable regularity. However, as constructing such global solutions in the large data case is still an open question, we will assume *a priori* that V_0 generates a global solution V in $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1}(\mathbb{R}^d))$. This only requirement will ensure, thanks to the result of Gallagher-Iftimie-Planchon in [10], that we have in fact a stronger property, namely

$$(1.7) \quad V \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1}(\mathbb{R}^d)) \quad \text{and} \quad V_t, \nabla^2 V \in L_1(\mathbb{R}_+; \dot{B}_{2,1}^{d/2+1}(\mathbb{R}^d)).$$

Theorem 1.2. *Assume that $d \geq 3$. Let $v_0 \in \dot{B}_{2,1}^{d/2-1}(\mathbb{R}^d)$ and ϱ_0 such that $a_0 := (\varrho_0 - 1) \in \dot{B}_{2,1}^{d/2-1} \cap \dot{B}_{2,1}^{d/2}(\mathbb{R}^d)$. Suppose that (1.3) with initial datum $V_0 := \mathcal{P}v_0$ generates a unique global solution $V \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})$ (thus also (1.7) is fulfilled), and denote*

$$M := \|V\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})} + \|V_t, \mu \nabla^2 V\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})}.$$

There exists a (large) universal constant C such that if ν satisfies

$$Ce^{CM}(\|a\|_{\dot{B}_{2,1}^{d/2-1}} + \nu\|a_0\|_{\dot{B}_{2,1}^{d/2}} + \|\mathcal{Q}u_0\|_{\dot{B}_{2,1}^{d/2-1}} + M^2 + \mu^2) \leq \sqrt{\nu}\sqrt{\mu},$$

then (1.1) has a unique global-in-time solution (ϱ, v) such that

$$(1.8) \quad \begin{aligned} v &\in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1}), \quad v_t, \nabla^2 v \in L_1(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1}), \\ a &:= (\varrho - 1) \in \mathcal{C}(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1} \cap \dot{B}_{2,1}^{d/2}) \cap L_2(\mathbb{R}_+; \dot{B}_{2,1}^{d/2}). \end{aligned}$$

In addition,

$$\begin{aligned} &\|\mathcal{Q}v\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})} + \|\mathcal{Q}v_t, \nu\nabla^2 \mathcal{Q}v\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})} + \|a\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})} + \nu\|a\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{d/2})} \\ &+ \nu^{1/2}(\|\mathcal{P}v - V\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})} + \|\mathcal{P}v_t - V_t, \mu\nabla^2(\mathcal{P}v - V)\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})}) \\ &\leq Ce^{CM}(\|a_0\|_{\dot{B}_{2,1}^{d/2-1}} + \nu\|a_0\|_{\dot{B}_{2,1}^{d/2}} + \|\mathcal{Q}u_0\|_{\dot{B}_{2,1}^{d/2-1}} + M^2 + \mu^2) \end{aligned}$$

and $(\varrho, v) \rightarrow (1, V)$ as follows:

$$\begin{aligned} &\|\varrho - 1\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{d/2})} + \|\nabla^2 \mathcal{Q}v\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})} + \|\mathcal{P}v - V\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})} \\ &+ \|\mathcal{P}v_t - V_t, \mu\nabla^2(\mathcal{P}v - V)\|_{L_1(\mathbb{R}_+; \dot{B}_{2,1}^{d/2-1})} \leq C\nu^{-1/2}\sqrt{\mu}. \end{aligned}$$

Let us emphasize that even in dimension $d \geq 3$, there are examples of large initial data for (1.3) generating global smooth solutions. One can refer for instance to [1, 3, 20, 25] and citations therein. Therefore, our second result indeed points out example of large data giving rise to global strong solutions for the compressible system (1.1).

Let us finally say a few words on our functional setting. Throughout, we used the so-called critical Besov spaces of type $\dot{B}_{2,1}^s(\mathbb{R}^2)$, as they are known to provide essentially the largest class of data for which System (1.1) may be solved by energy type methods, and is well-posed in the sense of Hadamard. As a matter of fact, our proof relies on a suitable energy method applied to the system after localization according to Littlewood-Paley decomposition (see the definition in the next section). We believe that it would be possible to derive similar qualitative results in the critical L_p Besov framework (spaces $\dot{B}_{p,1}^s$). However, we refrained from doing that both because it makes the proof significantly more technical, and because there are some restrictions to the admissible values of p (e.g. $2 \leq p < 4$ if $d = 2$) so that the improvement compared to $p = 2$ is not so big.

The rest of the paper unfolds as follows. In the next section, we introduce Besov spaces and recall basic facts about them. Section 3 is devoted to proving both Theorem 1.1 and 1.2. In fact, we are able to provide a common proof to both results as the only difference between dimension $d = 2$ and dimension $d \geq 3$ is that we are always guaranteed that V_0 gives rise to a global regular solution in the former case while it is an additional assumption in the latter case. In Appendix we show Theorem 4.1 concerning the regularity of 2D incompressible flow.

2. NOTATION, BESOV SPACES AND BASIC PROPERTIES

The *Littlewood-Paley decomposition* plays a central role in our analysis. To define it, fix some smooth radial non increasing function χ supported in the ball $B(0, \frac{4}{3})$ of \mathbb{R}^d , and with value 1 on, say, $B(0, \frac{3}{4})$, then set $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. We have

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1 \quad \text{in } \mathbb{R}^d \setminus \{0\} \quad \text{and} \quad \text{Supp } \varphi \subset \{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined on tempered distributions by

$$\dot{\Delta}_j u := \varphi(2^{-j} D)u := \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F}u) = 2^{jd} h(2^j \cdot) \star u \quad \text{with} \quad h := \mathcal{F}^{-1} \varphi.$$

In order to ensure that

$$(2.9) \quad f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d),$$

we restrict our attention to those tempered distributions f such that

$$(2.10) \quad \lim_{k \rightarrow -\infty} \|\dot{S}_k f\|_{L^\infty} = 0,$$

where $\dot{S}_k f$ stands for the low frequency cut-off defined by $\dot{S}_k f := \chi(2^{-k} D)f$.

Definition 2.1. For $s \in \mathbb{R}$ the homogeneous Besov space $\dot{B}_{2,1}^s := \dot{B}_{2,1}^s(\mathbb{R}^d)$ is the set of tempered distributions f satisfying (2.10) and

$$\|f\|_{\dot{B}_{2,1}^s} := \sum_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L_2} < \infty.$$

Remark 2.1. For $s \leq d/2$ (which is the only case we are concerned with in this paper), $\dot{B}_{2,1}^s$ is a Banach space which coincides with the completion for $\|\cdot\|_{\dot{B}_{2,1}^s}$ of the set $\mathcal{S}_0(\mathbb{R}^d)$ of Schwartz functions with Fourier transform supported away from the origin.

In many parts of the paper, it will be suitable to split tempered distributions f (e.g. the unknown $a := \varrho - 1$) into low and high frequencies as follows:

$$(2.11) \quad f^\ell := \sum_{2^k \nu \leq 1} \dot{\Delta}_k f \quad \text{and} \quad f^h := \sum_{2^k \nu > 1} \dot{\Delta}_k f.$$

The following Bernstein inequalities play an important role in our analysis:

- Direct Bernstein inequality: for all $1 \leq p \leq q \leq \infty$ and $k \in \mathbb{N}$,

$$(2.12) \quad \|\dot{\Delta}_j \nabla^k u\|_{L_q(\mathbb{R}^d)} \leq C 2^{j(k+d(\frac{1}{p}-\frac{1}{q}))} \|\dot{\Delta}_j u\|_{L_p(\mathbb{R}^d)}.$$

- Reverse Bernstein inequality: for all $1 \leq p \leq \infty$, we have

$$\|\dot{\Delta}_j u\|_{L_p(\mathbb{R}^d)} \leq C 2^{-j} \|\dot{\Delta}_j \nabla u\|_{L_p(\mathbb{R}^d)}.$$

The following lemma will be needed to estimate the nonlinear terms of (1.1) and (1.3). It is just a consequence of Bony decomposition and of continuity results for the paraproduct and remainder operators, as stated in e.g. Theorem 2.52 of [2].

Lemma 2.1. Let $g \in \dot{B}_{2,1}^{s_1}(\mathbb{R}^d)$ and $h \in \dot{B}_{2,1}^{s_2}(\mathbb{R}^d)$ for some couple (s_1, s_2) satisfying

$$s_1 \leq d/2, \quad s_2 \leq d/2 \quad \text{and} \quad s_1 + s_2 > 0.$$

Then $gh \in \dot{B}_{2,1}^{s_1+s_2-d/2}(\mathbb{R}^d)$, and we have

$$(2.13) \quad \|gh\|_{\dot{B}_{2,1}^{s_1+s_2-d/2}} \leq C \|g\|_{\dot{B}_{2,1}^{s_1}} \|h\|_{\dot{B}_{2,1}^{s_2}}.$$

Finally, let us recall that any vector field $w = (w^1, \dots, w^d)$ with components in $\mathcal{S}'(\mathbb{R}^d)$ satisfying (2.10) may be decomposed into one potential part $\mathcal{Q}w$ and one divergence-free part $\mathcal{P}w$, where the projectors \mathcal{P} and \mathcal{Q} are defined by

$$\mathcal{Q} := -(-\Delta)^{-1} \nabla \operatorname{div} \quad \text{and} \quad \mathcal{P} := \operatorname{Id} + (-\Delta)^{-1} \nabla \operatorname{div}.$$

In particular, because \mathcal{P} and \mathcal{Q} are smooth homogeneous of degree 0 Fourier multipliers, they map $\dot{B}_{2,1}^s(\mathbb{R}^d)$ to itself for any $s \leq d/2$.

3. THE PROOF OF THE MAIN RESULTS

We shall get Theorems 1.1 and 1.2 altogether. In fact, if it is known that the limit system (1.3) with initial data $V_0 := \mathcal{P}v_0$ has a unique solution in our functional setting, then the proof is the same in any dimension $d \geq 2$. The only difference is that in the 2D case the existence of a global solution to (1.3) is ensured by Theorem 4.1 for arbitrary large data whereas, if $d \geq 3$, it is indeed a supplementary assumption.

To simplify the presentation, we assume from now on that the shear viscosity μ is 1. This is not restrictive owing to the following change of unknowns and volume viscosity:

$$(3.14) \quad (\tilde{\varrho}, \tilde{v})(t, x) := (\varrho, v)(\mu t, \mu x) \quad \text{and} \quad \tilde{\lambda} = \lambda/\mu.$$

We concentrate our attention on the proof of *global in time* a priori estimates, as the local existence issue is nowadays well understood. For example, just assuming that ϱ_0 is bounded away from zero and that the regularity assumptions of Theorems 1.1 or 1.2 are fulfilled, Theorem 2 of [7] provides us with a unique local solution (ϱ, v) to (1.1) such that

$$(3.15) \quad a := (\varrho - 1) \in \mathcal{C}([0, T]; \dot{B}_{2,1}^{d/2}) \quad \text{and} \quad v \in \mathcal{C}([0, T]; \dot{B}_{2,1}^{d/2-1}) \cap L_1(0, T; \dot{B}_{2,1}^{d/2+1}).$$

Furthermore, continuation beyond T is possible if

$$(3.16) \quad \int_0^T \|\nabla v\|_{L_\infty} dt < \infty, \quad \|a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2})} < \infty \quad \text{and} \quad \inf_{(t,x) \in [0, T] \times \mathbb{R}^d} \varrho(t, x) > 0.$$

Finally, from standard results for the transport equation, we see that the additional $\dot{B}_{2,1}^{d/2-1}$ regularity of a is preserved through the evolution.

Next, to compare the solutions of (1.1) and (1.3), we set $u := v - V$. From the very beginning, the potential $\mathcal{Q}u$ and divergence-free $\mathcal{P}u$ parts of the perturbation of the velocity are treated separately. On one hand, because $\mathcal{Q}u = \mathcal{Q}v$, applying \mathcal{Q} to the velocity equation in (1.1) yields

$$(\mathcal{Q}u)_t + \mathcal{Q}((1+a)v \cdot \nabla v) - \nu \Delta \mathcal{Q}u + P'(1+a)\nabla a = -\mathcal{Q}(av_t),$$

hence using $v = \mathcal{Q}u + \mathcal{P}u + V$ and assuming $P'(1) = 1$ (for notational simplicity),

$$(3.17) \quad (\mathcal{Q}u)_t + \mathcal{Q}((u+V) \cdot \nabla \mathcal{Q}u) - \nu \Delta \mathcal{Q}u + \nabla a = -\mathcal{Q}(aV_t + au_t) - \mathcal{Q}R_2$$

with, denoting $k(a) := P'(1+a) - P'(1) = P'(1+a) - 1$,

$$(3.18) \quad R_2 = (1+a)(u+V) \cdot \nabla \mathcal{P}u + (1+a)(u+V) \cdot \nabla V + a(u+V) \cdot \nabla \mathcal{Q}u + k(a)\nabla a,$$

and a satisfying

$$(3.19) \quad a_t + \operatorname{div}(au) + \operatorname{div} \mathcal{Q}u + V \cdot \nabla a = 0.$$

Initial data are $\mathcal{Q}u|_{t=0} = \mathcal{Q}v_0$ and $a|_{t=0} = a_0$.

On the other hand, applying \mathcal{P} to the velocity equation of (1.1) and subtracting the equation for $\mathcal{P}V = V$ in (1.3), we discover that

$$\begin{aligned} (\mathcal{P}u)_t + \mathcal{P}((u+V) \cdot \nabla \mathcal{P}u) - \Delta \mathcal{P}u &= -\mathcal{P}(aV_t + au_t) \\ &\quad - \mathcal{P}\left((1+a)(u+V) \cdot \nabla \mathcal{Q}u + (1+a)u \cdot \nabla V + a(u+V) \cdot \nabla \mathcal{P}u + aV \cdot \nabla V\right), \end{aligned}$$

supplemented with the initial datum $\mathcal{P}u|_{t=0} = 0$ (as we assumed $V_0 = \mathcal{P}v_0$).

Note that because $\mathcal{Q}u \cdot \nabla \mathcal{Q}u$ is a gradient, one may rewrite the above equation as

$$(3.20) \quad (\mathcal{P}u)_t + \mathcal{P}((u + V) \cdot \nabla \mathcal{P}u) - \Delta \mathcal{P}u = -\mathcal{P}(aV_t + au_t) - \mathcal{P}R_1$$

with

$$(3.21) \quad R_1 := (1 + a)\mathcal{P}u \cdot \nabla(V + \mathcal{Q}u) + (1 + a)V \cdot \nabla \mathcal{Q}u + (1 + a)\mathcal{Q}u \cdot \nabla V \\ + a(u + V) \cdot \nabla \mathcal{P}u + aV \cdot \nabla V + a\mathcal{Q}u \cdot \nabla \mathcal{Q}u.$$

Let us start the computations. The general approach is adapted from [4]: we localize equations (3.17), (3.19) and (3.20) in the frequency space by means of the dyadic operators $\dot{\Delta}_j$, and perform suitable energy estimates. The key point is that we strive for time pointwise estimates of u , a and $\nu \nabla a$ in the same (Besov) space.

In what follows, we denote by a^ℓ and a^h the low and high frequencies parts of a , respectively (see (2.11)) and set

$$X_d(T) := \|\mathcal{Q}u, a, \nu \nabla a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2-1})}, \quad Y_d(T) := \|\mathcal{Q}u_t, \nu \nabla^2 \mathcal{Q}u, \nu \nabla^2 a^\ell, \nabla a^h\|_{L_1(0, T; \dot{B}_{2,1}^{d/2-1})}, \\ Z_d(T) := \|\mathcal{P}u\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2-1})}, \quad W_d(T) := \|\mathcal{P}u_t, \nabla^2 \mathcal{P}u\|_{L_1(0, T; \dot{B}_{2,1}^{d/2-1})}.$$

We assume that the maximal solution $(\varrho = 1 + a, v)$ of (1.1) corresponding to data (ϱ_0, v_0) is defined on the time interval $[0, T_\star)$ and satisfies (3.15), and we fix some $M \geq 0$ so that the ‘incompressible solution’ V to (1.3) fulfills

$$(3.22) \quad V_d(T) := \|V\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2-1})} + \|V_t, \nabla^2 V\|_{L_1(0, T; \dot{B}_{2,1}^{d/2-1})} \leq M \quad \text{for all } T \geq 0.$$

As already pointed out and proved in Appendix, in the 2D case, number M may be expressed in terms of $\|V_0\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)}$.

We claim that if ν is large enough then one may find some (large) D and (small) δ so that for all $T < T_\star$, the following bounds are valid:

$$(3.23) \quad X_d(T) + Y_d(T) \leq D \quad \text{and} \quad Z_d(T) + W_d(T) \leq \delta.$$

Step 1. Estimates for the divergence-free part of the velocity. Applying $\dot{\Delta}_j$ to (3.20), taking the L_2 inner product with $\dot{\Delta}_j \mathcal{P}u$ then using that $\mathcal{P}^2 = \mathcal{P}$, we discover that

$$\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_j \mathcal{P}u\|_{L_2}^2 + \|\nabla \dot{\Delta}_j \mathcal{P}u\|_{L_2}^2 + \int (\dot{\Delta}_j(u + V) \cdot \nabla \mathcal{P}u) \cdot \dot{\Delta}_j \mathcal{P}u \, dx \\ = - \int \dot{\Delta}_j(aV_t + au_t + R_1) \cdot \dot{\Delta}_j \mathcal{P}u \, dx.$$

Then using Bernstein’s inequalities in the second term, swapping operators $\dot{\Delta}_j$ and $u + V$ in the third term, and integrating by parts, we get we get for some universal constant $c > 0$,

$$(3.24) \quad \frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_j \mathcal{P}u\|_{L_2}^2 + c \|\nabla^2 \dot{\Delta}_j \mathcal{P}u\|_{L_2} \|\dot{\Delta}_j \mathcal{P}u\|_{L_2} \leq \frac{1}{2} \int |\dot{\Delta}_j \mathcal{P}u|^2 \operatorname{div} u \, dx \\ + \int ([u + V, \dot{\Delta}_j] \cdot \nabla \mathcal{P}u) \cdot \dot{\Delta}_j \mathcal{P}u \, dx - \int \dot{\Delta}_j(aV_t + au_t + R_1) \cdot \dot{\Delta}_j \mathcal{P}u \, dx.$$

It is well known (see e.g. Lemma 2.100 in [2]) that the commutator term may be estimated as follows:

$$(3.25) \quad 2^{j(d/2-1)} \|[u + V, \dot{\Delta}_j] \cdot \nabla \mathcal{P}u\|_{L_2} \leq C c_j \|\nabla(u + V)\|_{\dot{B}_{2,1}^{d/2}} \|\mathcal{P}u\|_{\dot{B}_{2,1}^{d/2-1}} \quad \text{with} \quad \sum_{j \in \mathbb{Z}} c_j = 1.$$

Hence dividing (formally) (3.24) by $\|\dot{\Delta}_j \mathcal{P}u\|_{L_2}$, multiplying by $2^{j(d/2-1)}$, integrating (3.24), remembering that $\mathcal{P}u|_{t=0} = 0$ and summing over j , we obtain

$$(3.26) \quad \|\mathcal{P}u\|_{L_\infty(0,T;\dot{B}_{2,1}^{d/2-1})} + \|\nabla^2 \mathcal{P}u\|_{L_1(0,T;\dot{B}_{2,1}^{d/2-1})} \\ \lesssim \int_0^T \|\nabla(u+V)\|_{\dot{B}_{2,1}^{d/2}} \|\mathcal{P}u\|_{\dot{B}_{2,1}^{d/2-1}} dt + \int_0^T \|aV_t + au_t\|_{\dot{B}_{2,1}^{d/2-1}} dt + \int_0^T \|R_1\|_{\dot{B}_{2,1}^{d/2-1}} dt.$$

Next we see, thanks to Lemma 2.1 that

$$\|aV_t + au_t\|_{L_1(0,T;\dot{B}_{2,1}^{d/2-1})} \lesssim \nu^{-1} \|\mathcal{Q}u_t, \mathcal{P}u_t, V_t\|_{L_1(0,T;\dot{B}_{2,1}^{d/2-1})} \|\nu a\|_{L_\infty(0,T;\dot{B}_{2,1}^{d/2})} \\ \lesssim \nu^{-1} (Y_d(T) + W_d(T) + V_d(T)) X_d(T).$$

In order to bound R_1 , we use the fact that

$$\|(1+a)\mathcal{P}u \cdot \nabla(V + \mathcal{Q}u)\|_{\dot{B}_{2,1}^{d/2-1}} \lesssim (1 + \|a\|_{\dot{B}_{2,1}^{d/2}}) \|\nabla(V + \mathcal{Q}u)\|_{\dot{B}_{2,1}^{d/2}} \|\mathcal{P}u\|_{\dot{B}_{2,1}^{d/2-1}}$$

and

$$\|(1+a)(\mathcal{Q}u \cdot \nabla V + V \cdot \nabla \mathcal{Q}u)\|_{\dot{B}_{2,1}^{d/2-1}} \lesssim (1 + \|a\|_{\dot{B}_{2,1}^{d/2}}) \|\mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2}} \|V\|_{\dot{B}_{2,1}^{d/2}},$$

whence, integrating on $[0, T]$ and using the interpolation inequality

$$\|z\|_{\dot{B}_{2,1}^{d/2}} \leq C \|z\|_{\dot{B}_{2,1}^{d/2-1}}^{1/2} \|\nabla^2 z\|_{\dot{B}_{2,1}^{d/2-1}}^{1/2} \quad \text{for } z = V, \mathcal{Q}u,$$

we get

$$\|(1+a)(\mathcal{Q}u \cdot \nabla V + V \cdot \nabla \mathcal{Q}u)\|_{L_1(0,T;\dot{B}_{2,1}^{d/2-1})} \lesssim (1 + \nu^{-1} X_d(T)) \nu^{-1/2} X_d(T)^{1/2} Y_d^{1/2}(T) V_d(T).$$

Finally, we have

$$\|a(u+V) \cdot \nabla \mathcal{P}u\|_{L_1(0,T;\dot{B}_{2,1}^{d/2-1})} \lesssim \|a\|_{L_\infty(0,T;\dot{B}_{2,1}^{d/2})} \|u+V\|_{L_\infty(0,T;\dot{B}_{2,1}^{d/2-1})} \|\nabla \mathcal{P}u\|_{L_1(0,T;\dot{B}_{2,1}^{d/2})} \\ \lesssim \nu^{-1} X_d(T) (Z_d(T) + X_d(T) + V_d(T)) W_d(T),$$

and

$$\|aV \cdot \nabla V + a\mathcal{Q}u \cdot \nabla \mathcal{Q}u\|_{L_1(0,T;\dot{B}_{2,1}^{d/2-1})} \lesssim \|a\|_{L_\infty(0,T;\dot{B}_{2,1}^{d/2})} (\|V\|_{L_\infty(0,T;\dot{B}_{2,1}^{d/2-1})} \|\nabla V\|_{L_1(0,T;\dot{B}_{2,1}^{d/2})} \\ + \|\mathcal{Q}u\|_{L_\infty(0,T;\dot{B}_{2,1}^{d/2-1})} \|\nabla \mathcal{Q}u\|_{L_1(0,T;\dot{B}_{2,1}^{d/2})}).$$

Therefore, we obtain

$$Z_d(T) + W_d(T) \lesssim \int_0^T \|\nabla V, \nabla \mathcal{P}u, \nabla \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2}} Z_d(t) dt \\ + \nu^{-1} (Y_d(T) + W_d(T) + V_d(T)) X_d(T) + \nu^{-1/2} X_d(T)^{1/2} Y_d^{1/2}(T) V_d(T) (1 + \nu^{-1} X_d(T)) \\ + \nu^{-1} X_d(T) (Z_d(T) + X_d(T) + V_d(T)) W_d(T) + \nu^{-1} X_d(T) (V_d^2(T) + \nu^{-1} X_d(T) Y_d(T)).$$

Assuming from now on that

$$(3.27) \quad X_d(T) \ll \nu,$$

and using Gronwall lemma, we conclude that

$$(3.28) \quad \begin{aligned} Z_d(T) + W_d(T) &\leq C e^{C \int_0^t \|\nabla V, \nabla \mathcal{P}u, \nabla \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2}} d\tau} \left(\nu^{-1} (Y_d(T) + V_d(T) + W_d(T)) X_d(T) \right. \\ &\quad \left. + \nu^{-1/2} X_d(T)^{1/2} Y_d^{1/2}(T) V_d(T) \right. \\ &\quad \left. + \nu^{-1} X_d(T) (Z_d(T) + X_d(T) + V_d(T)) W_d(T) + \nu^{-1} X_d(T) V_d^2(T) \right). \end{aligned}$$

Step 2. Estimate on the potential part of the velocity and on the density. To estimate the potential part of the velocity, we are required to consider the momentum and continuity equations altogether. Now, localizing (3.19) and (3.17) according to Littlewood-Paley operators, we discover that

$$(3.29) \quad a_{j,t} + (u + V) \cdot \nabla a_j + \operatorname{div} \mathcal{Q}u_j = g_j$$

$$(3.30) \quad \mathcal{Q}u_{j,t} + \mathcal{Q}((u + V) \cdot \nabla \mathcal{Q}u_j) - \nu \Delta \mathcal{Q}u_j + \nabla a_j = f_j$$

where

$$(3.31) \quad a_j := \dot{\Delta}_j a, \quad \mathcal{Q}u_j := \dot{\Delta}_j \mathcal{Q}u,$$

$$(3.32) \quad g_j := -\dot{\Delta}_j (a \operatorname{div} \mathcal{Q}u) - [\dot{\Delta}_j, (u + V)] \cdot \nabla a$$

and

$$(3.33) \quad f_j := -\dot{\Delta}_j \mathcal{Q}(a V_t + a u_t) - \dot{\Delta}_j \mathcal{Q}R_2 - [\dot{\Delta}_j, u + V] \cdot \nabla \mathcal{Q}u.$$

We follow an energy method to bound each term $(a_j, \mathcal{Q}u_j)$. More precisely, testing (3.29) and (3.30) by a_j and $\mathcal{Q}u_j$, respectively, yields

$$(3.34) \quad \frac{1}{2} \frac{d}{dt} \int a_j^2 dx + \int a_j \operatorname{div} \mathcal{Q}u_j dx = \frac{1}{2} \int \operatorname{div} u a_j^2 dx + \int g_j a_j dx$$

and

$$(3.35) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\mathcal{Q}u_j|^2 dx + \nu \int |\nabla \mathcal{Q}u_j|^2 dx - \int a_j \operatorname{div} \mathcal{Q}u_j dx \\ = \frac{1}{2} \int \operatorname{div} u |\mathcal{Q}u_j|^2 dx + \int f_j \cdot \mathcal{Q}u_j dx. \end{aligned}$$

We next want an estimate for $\|\nabla a_j\|_{L^2}^2$. From (3.29), we have

$$(3.36) \quad \nabla a_{j,t} + (u + V) \cdot \nabla \nabla a_j + \nabla \operatorname{div} \mathcal{Q}u_j = \nabla g_j - \nabla(u + V) \cdot \nabla a_j.$$

Testing that equation by ∇a_j yields

$$(3.37) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\nabla a_j|^2 dx + \int ((u + V) \cdot \nabla \nabla a_j) \cdot \nabla a_j dx + \int \nabla \operatorname{div} \mathcal{Q}u_j \cdot \nabla a_j dx \\ = \int (\nabla g_j - \nabla(u + V) \cdot \nabla a_j) \cdot \nabla a_j dx. \end{aligned}$$

To eliminate the highest order term, namely the one with $\nabla \operatorname{div} \mathcal{Q}u_j$, it is suitable to combine the above equality with a relation involving $\int \mathcal{Q}u_j \cdot \nabla a_j dx$. Now, testing (3.36) by

$\mathcal{Q}u_j$ and the momentum equation by ∇a_j , we get

$$(3.38) \quad \frac{d}{dt} \int \mathcal{Q}u_j \cdot \nabla a_j \, dx + \int (u + V) \cdot \nabla (\mathcal{Q}u_j \cdot \nabla a_j) \, dx - \nu \int \Delta \mathcal{Q}u_j \cdot \nabla a_j \, dx \\ + \int |\nabla a_j|^2 \, dx + \int \nabla \operatorname{div} \mathcal{Q}u_j \cdot \mathcal{Q}u_j \, dx = \int (\nabla g_j - \nabla(u + V) \cdot \nabla a_j) \cdot \mathcal{Q}u_j \, dx + \int f_j \cdot \nabla a_j \, dx.$$

Note that by integration by parts, we have

$$\int (u + V) \cdot \nabla (\mathcal{Q}u_j \cdot \nabla a_j) \, dx = - \int \mathcal{Q}u_j \cdot \nabla a_j \operatorname{div} u \, dx.$$

Hence adding ν times (3.37) to (3.38) and noting that $\Delta \mathcal{Q}u_j \equiv \nabla \operatorname{div} \mathcal{Q}u_j$, the highest order terms cancel out, and we get

$$\frac{1}{2} \frac{d}{dt} \int (\nu |\nabla a_j|^2 + 2 \mathcal{Q}u_j \cdot \nabla a_j) \, dx + \nu \int (|\nabla a_j|^2 - |\Delta \mathcal{Q}u_j|^2) \, dx \\ = \int \left(\frac{\nu}{2} |\nabla a_j|^2 + \mathcal{Q}u_j \cdot \nabla a_j \right) \operatorname{div} u \, dx + \nu \int (\nabla g_j - \nabla(u + V) \cdot \nabla a_j) \cdot \nabla a_j \, dx \\ + \int (\nabla g_j - \nabla(u + V) \cdot \nabla a_j) \cdot \mathcal{Q}u_j \, dx + \int f_j \cdot \nabla a_j \, dx.$$

After multiplying the above equality by ν and adding up twice (3.34) and (3.35), we get

$$(3.39) \quad \frac{1}{2} \frac{d}{dt} \mathcal{L}_j^2 + \nu \int (|\nabla \mathcal{Q}u_j|^2 + |\nabla a_j|^2) \, dx \\ = \int (2g_j a_j + 2f_j \cdot \mathcal{Q}u_j + \nu^2 \nabla g_j \cdot \nabla a_j + \nu \nabla g_j \cdot \mathcal{Q}u_j + \nu f_j \cdot \nabla a_j) \, dx \\ + \frac{1}{2} \int \mathcal{L}_j^2 \operatorname{div} u \, dx - \nu \int (\nabla(u + V) \cdot \nabla a_j) \cdot (\nu \nabla a_j + \mathcal{Q}u_j) \, dx$$

with

$$(3.40) \quad \mathcal{L}_j^2 := \int (2a_j^2 + 2|\mathcal{Q}u_j|^2 + 2\nu \mathcal{Q}u_j \cdot \nabla a_j + |\nu \nabla a_j|^2) \, dx.$$

At this stage, two fundamental observations are in order. First, we obviously have

$$(3.41) \quad \mathcal{L}_j \approx \|(\mathcal{Q}u_j, a_j, \nu \nabla a_j)\|_{L_2} \quad \text{for all } j \in \mathbb{Z}$$

and, second,

$$(3.42) \quad \nu \int (|\nabla \mathcal{Q}u_j|^2 + |\nabla a_j|^2) \, dx \geq c \min(\nu 2^{2j}, \nu^{-1}) \mathcal{L}_j^2.$$

Therefore (3.39), (3.41) and (3.42) lead to

$$\frac{1}{2} \frac{d}{dt} \mathcal{L}_j^2 + c \min(\nu 2^{2j}, \nu^{-1}) \mathcal{L}_j^2 \leq \left(\frac{1}{2} \|\operatorname{div} u\|_{L_\infty} + C \|\nabla(u + V)\|_{L_\infty} \right) \mathcal{L}_j^2 + C \| [g_j, f_j, \nu \nabla g_j] \|_{L_2} \mathcal{L}_j,$$

whence integrating in time,

$$(3.43) \quad \mathcal{L}_j(t) + c \min(\nu 2^{2j}, \nu^{-1}) \int_0^t \mathcal{L}_j \, d\tau \\ \leq \mathcal{L}_j(0) + C \int_0^t \|\nabla(u + V)\|_{L_\infty} \mathcal{L}_j \, d\tau + C \int_0^t \| [g_j, f_j, \nu \nabla g_j] \|_{L_2} \, d\tau.$$

Note that we lost the expected parabolic smoothing of $\mathcal{Q}u$ because $\min(\nu 2^{2j}, \nu^{-1}) = \nu^{-1}$ for large j 's. However, it may be recovered by starting directly from (3.35) and integrating by parts in the term with $a_j \operatorname{div} \mathcal{Q}u_j$. After using Bernstein and Hölder inequalities, we arrive at

$$\frac{1}{2} \frac{d}{dt} \|\mathcal{Q}u_j\|_{L_2}^2 + c\nu 2^{2j} \|\mathcal{Q}u_j\|_{L_2}^2 \leq \|\nabla a_j\|_{L_2} \|\mathcal{Q}u_j\|_{L_2} + \frac{1}{2} \|\operatorname{div} u\|_{L_\infty} \|\mathcal{Q}u_j\|_{L_2}^2 + \|f_j\|_{L_2} \|\mathcal{Q}u_j\|_{L_2},$$

whence, integrating in time,

$$\begin{aligned} \|\mathcal{Q}u_j(t)\|_{L_2} + c\nu 2^{2j} \int_0^t \|\mathcal{Q}u_j\|_{L_2} d\tau &\leq \|\mathcal{Q}u_j(0)\|_{L_2} \\ &\quad + \int_0^t \|\nabla a_j\|_{L_2} d\tau + \frac{1}{2} \int_0^t \|\operatorname{div} u\|_{L_\infty} \|\mathcal{Q}u_j\|_{L_2} d\tau + \int_0^t \|f_j\|_{L_2} d\tau. \end{aligned}$$

Putting together with (3.43), remembering (3.41), multiplying by $2^{j(d/2-1)}$ and eventually summing up over $j \in \mathbb{Z}$, we end up with the following fundamental inequality:

$$\begin{aligned} (3.44) \quad \|(a, \nu \nabla a, u)(t)\|_{\dot{B}_{2,1}^{d/2-1}} + \nu \int_0^t \|\nabla a^\ell, \nabla \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2}} d\tau + \int_0^t \|a^h\|_{\dot{B}_{2,1}^{d/2}} d\tau \\ \lesssim \|(a, \nu \nabla a, u)(0)\|_{\dot{B}_{2,1}^{d/2-1}} + \int_0^t \|\nabla(u+V)\|_{L_\infty} \|(a, \nu \nabla a, \mathcal{Q}u)\|_{\dot{B}_{2,1}^{d/2-1}} d\tau \\ + \int_0^t \sum_{j \in \mathbb{Z}} 2^{j(d/2-1)} \|[g_j, f_j, \nu \nabla g_j]\|_{L_2} d\tau, \end{aligned}$$

where notations a^ℓ and a^h have been defined in (2.11).

To complete the proof of estimates for a and $\mathcal{Q}u$, we now have to get suitable bounds for the last term in (3.44). Let us start with the study of g_j defined by (3.32). First we see that, by virtue of Lemma 2.1, we have

$$(3.45) \quad \|a \operatorname{div} \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2-1}} \lesssim \|\operatorname{div} \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2}} \|a\|_{\dot{B}_{2,1}^{d/2-1}}$$

and, using rule and, again, Lemma 2.1,

$$(3.46) \quad \nu \|\nabla(a \operatorname{div} \mathcal{Q}u)\|_{\dot{B}_{2,1}^{d/2-1}} \lesssim \|\nabla \operatorname{div} \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2-1}} \|\nu a\|_{\dot{B}_{2,1}^{d/2}} + \|\operatorname{div} \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2}} \|\nu \nabla a\|_{\dot{B}_{2,1}^{d/2-1}}.$$

The commutator term may be bounded as follows (the first inequality stems from Lemma 2.100 in [2], and the second one may be deduced from that lemma and Leibniz rule):

$$(3.47) \quad \sum_{j \in \mathbb{Z}} 2^{j(d/2-1)} \|[\dot{\Delta}_j, (u+V)] \nabla a\|_{L_2} \leq C \|\nabla(u+V)\|_{\dot{B}_{2,1}^{d/2}} \|a\|_{\dot{B}_{2,1}^{d/2-1}},$$

$$(3.48) \quad \sum_{j \in \mathbb{Z}} 2^{j(d/2-1)} \nu \|\nabla([\dot{\Delta}_j, (u+V)] \nabla a)\|_{L_2} \leq C \|\nabla(u+V)\|_{\dot{B}_{2,1}^{d/2}} \|\nu \nabla a\|_{\dot{B}_{2,1}^{d/2-1}}.$$

Hence, putting (3.45) to (3.48) together, we get

$$(3.49) \quad \sum_{j \in \mathbb{Z}} 2^{j(d/2-1)} \|g_j, \nu \nabla g_j\|_{L_2} \leq C \|\nabla(u+V)\|_{\dot{B}_{2,1}^{d/2}} (\|a\|_{\dot{B}_{2,1}^{d/2-1}} + \nu \|a\|_{\dot{B}_{2,1}^{d/2}}).$$

Next, let us bound f_j defined in (3.33). To handle the terms corresponding to R_2 (see (3.18)), we use the fact that

$$(3.50) \quad \|(1+a)(u+V) \cdot \nabla(\mathcal{P}u+V)\|_{\dot{B}_{2,1}^{d/2-1}} \lesssim (1 + \|a\|_{\dot{B}_{2,1}^{d/2}}) \|(u, V)\|_{\dot{B}_{2,1}^{d/2-1}} \|(\nabla \mathcal{P}u, \nabla V)\|_{\dot{B}_{2,1}^{d/2}},$$

$$(3.51) \quad \|a(u+V) \cdot \nabla \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2-1}} \lesssim \|a\|_{\dot{B}_{2,1}^{d/2}} \|(u, V)\|_{\dot{B}_{2,1}^{d/2-1}} \|\nabla \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2-1}},$$

and (see (3.17))

$$(3.52) \quad \|k(a)\nabla a\|_{\dot{B}_{2,1}^{d/2-1}} \lesssim \|a\|_{\dot{B}_{2,1}^{d/2}}^2.$$

As for g , the commutator term of f may be bounded according to Lemma 2.100 in [2]:

$$(3.53) \quad \sum_{j \in \mathbb{Z}} 2^{j(d/2-1)} \|[\dot{\Delta}_j, u + V] \nabla \mathcal{Q}u\|_{L_2} \leq C \|\nabla(u + V)\|_{\dot{B}_{2,1}^{d/2}} \|\mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2-1}}.$$

Finally, for the terms with the time derivative, we have

$$(3.54) \quad \|aV_t\|_{\dot{B}_{2,1}^{d/2-1}} + \|au_t\|_{\dot{B}_{2,1}^{d/2-1}} \leq C \|V_t, u_t\|_{\dot{B}_{2,1}^{d/2-1}} \|a\|_{\dot{B}_{2,1}^{d/2}}.$$

From (3.50) to (3.54), we conclude that

$$(3.55) \quad \sum_{j \in \mathbb{Z}} 2^{j(d/2-1)} \|f_j\|_{L_2} \leq C \left(\|(u, V)\|_{\dot{B}_{2,1}^{d/2-1}} \|\nabla \mathcal{P}u, \nabla V\|_{\dot{B}_{2,1}^{d/2}} \right. \\ \left. + \|a\|_{\dot{B}_{2,1}^{d/2}} \left(\|(u, V)\|_{\dot{B}_{2,1}^{d/2-1}} \|\nabla u, \nabla V\|_{\dot{B}_{2,1}^{d/2}} + \|V_t, u_t\|_{\dot{B}_{2,1}^{d/2-1}} + \|a\|_{\dot{B}_{2,1}^{d/2}} \right) \right).$$

Putting (3.49) and (3.55) together in (3.44) gives us for all $0 \leq T < T_\star$,

$$(3.56) \quad \|\mathcal{Q}u, a, \nu \nabla a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2-1})} + \|\mathcal{Q}u_t, \nu \nabla^2 \mathcal{Q}u, \nu \nabla^2 a^\ell, \nabla a^h\|_{L_1(0, T; \dot{B}_{2,1}^{d/2-1})} \\ \lesssim \|(\mathcal{Q}u, a, \nu \nabla a)(0)\|_{\dot{B}_{2,1}^{d/2-1}} + \int_0^T \|\nabla u, \nabla V\|_{\dot{B}_{2,1}^{d/2}} \|a, \nu \nabla a, \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2-1}} d\tau \\ + (1 + \|a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2})}) \int_0^T \|(\mathcal{P}u, \mathcal{Q}u, V)\|_{\dot{B}_{2,1}^{d/2-1}} \|(\nabla \mathcal{P}u, \nabla V)\|_{\dot{B}_{2,1}^{d/2}} d\tau \\ + \|a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2})} \int_0^T \|(\mathcal{P}u, \mathcal{Q}u, V)\|_{\dot{B}_{2,1}^{d/2-1}} \|\nabla \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2}} d\tau \\ + \|V_t, u_t\|_{L_1(0, T; \dot{B}_{2,1}^{d/2-1})} \|a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2})} + \|a\|_{L_2(0, T; \dot{B}_{2,1}^{d/2})}^2.$$

Hence, using obvious interpolation to bound the last term, and also the fact that (3.27) implies that

$$(3.57) \quad \|a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2})} \ll 1,$$

we get

$$(3.58) \quad \|\mathcal{Q}u, a, \nu \nabla a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2-1})} + \|\mathcal{Q}u_t, \nu \nabla^2 \mathcal{Q}u, \nu \nabla^2 a^\ell, \nabla a^h\|_{L_1(0, T; \dot{B}_{2,1}^{d/2-1})} \\ \lesssim \|(\mathcal{Q}u, a, \nu \nabla a)(0)\|_{\dot{B}_{2,1}^{d/2-1}} + \int_0^t \|\nabla \mathcal{P}u, \nabla \mathcal{Q}u, \nabla V\|_{\dot{B}_{2,1}^{d/2}} \|a, \nu \nabla a, \mathcal{Q}u\|_{\dot{B}_{2,1}^{d/2-1}} d\tau \\ + \|(\mathcal{P}u, V)\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2-1})} \|(\nabla \mathcal{P}u, \nabla V)\|_{L_1(0, T; \dot{B}_{2,1}^{d/2})} \\ + \|a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2})} \|(\mathcal{P}u, V)\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2-1})} \|\nabla \mathcal{Q}u\|_{L_1(0, T; \dot{B}_{2,1}^{d/2})} \\ + \|V_t, \mathcal{P}u_t\|_{L_1(0, T; \dot{B}_{2,1}^{d/2-1})} \|a\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2})} + \nu^{-1} (\|a^\ell\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2-1})} \|\nu \nabla a^\ell\|_{L_1(0, T; \dot{B}_{2,1}^{d/2})} \\ + \|\nu a^h\|_{L_\infty(0, T; \dot{B}_{2,1}^{d/2})} \|a^h\|_{L_1(0, T; \dot{B}_{2,1}^{d/2})}).$$

Hence, from Gronwall lemma,

$$(3.59) \quad \begin{aligned} X_d(T) + Y_d(T) \leq & C e^{C \int_0^t \|\nabla \mathcal{P}u, \nabla \mathcal{Q}u, \nabla V\|_{\dot{B}_{2,1}^{d/2}} d\tau} \left(X_d(0) + (V_d(T) + Z_d(T))(V_d(T) + W_d(T)) \right. \\ & \left. + \nu^{-2} X_d(T) Y_d(T) (V_d(T) + Z_d(T)) + \nu^{-1} (V_d(T) + Y_d(T) + W_d(T)) X_d(T) \right). \end{aligned}$$

Step 3. Global-in-time closure of the estimates. Assuming that

$$(3.60) \quad \nu^{-1} D \ll 1,$$

Inequality (3.59) and hypotheses (3.22) and (3.23) imply that

$$\begin{aligned} X_d(T) + Y_d(T) \leq & C e^{C(M+\nu^{-1}D+\delta)} \left(X_d(0) + (M+\delta)^2 \right. \\ & \left. + \nu^{-2} D(M+\delta) X_d(T) + \nu^{-1} (D+\delta+M) X_d(T) \right) \end{aligned}$$

while (3.28) yields

$$\begin{aligned} Z_d(T) + W_d(T) \leq & C D e^{C(M+\nu^{-1}D+\delta)} \left(\nu^{-1} (M+\delta+D) \right. \\ & \left. + \nu^{-1/2} M + \nu^{-1} (M+D+\delta) W_d(T) + \nu^{-1} M^2 \right). \end{aligned}$$

Hence, assuming in addition that

$$(3.61) \quad \nu^{-1} D \leq M \quad \text{and} \quad \delta \leq \max\{M, 1\},$$

we get (enlarging C as the case may be)

$$\begin{aligned} X_d(T) + Y_d(T) &\leq C e^{CM} \left(X_d(0) + M^2 + 1 + \nu^{-1} (M+D) X_d(T) \right), \\ Z_d(T) + W_d(T) &\leq C D e^{CM} \left(\nu^{-1} D + \nu^{-1/2} M + \nu^{-1} M^2 + \nu^{-1} (M+D) W_d(T) \right). \end{aligned}$$

Therefore, if we make the assumption that

$$(3.62) \quad D(D+M+1) e^{CM} \ll \nu,$$

then we end up with

$$(3.63) \quad X_d(T) + Y_d(T) \leq C e^{CM} (X_d(0) + M^2 + 1)$$

and

$$(3.64) \quad Z_d(T) + W_d(T) \leq C D e^{CM} (\nu^{-1/2} M + \nu^{-1} (D + M^2 + 1)).$$

So it is natural to take first

$$(3.65) \quad D := C e^{CM} (X_d(0) + M^2 + 1)$$

and then to set

$$(3.66) \quad \delta := C e^{CM} (X_d(0) + M^2) \left(\nu^{-1/2} M + \nu^{-1} (X_d(0) + M^2 + 1) \right).$$

Now, assuming that for a suitably large (universal) constant C we have

$$(3.67) \quad C e^{CM} (X_d(0) + 1 + M^2) \leq \sqrt{\nu},$$

we see that Conditions (3.60) and (3.62) are fulfilled (and thus also (3.61) as it is weaker).

Let us recap: if ν and the compressible part of the data fulfill (3.67) then defining D and δ according to (3.65) and (3.66) ensures that (3.23) is fulfilled for all $T < T_*$. Then, combining with the continuation criterion recalled in (3.16), one can conclude that $T_* = +\infty$ and that (3.23) is satisfied for all time. This completes the proof of Theorems 1.1 and 1.2. Finally, in the 2D case, Theorem 4.1 enables us to take

$$(3.68) \quad M = C \|\mathcal{P}v_0\|_{\dot{B}_{2,1}^0} \exp\left(\frac{C}{\mu^4} \|\mathcal{P}v_0\|_{L_2}^4\right),$$

which provides us with an explicit lower bound for ν depending only on the initial data, through (3.67).

4. APPENDIX

We here consider the global well-posedness issue of the incompressible two-dimensional Navier-Stokes system in the critical Besov spaces setting. Although essentially the same result has been proved in [5] (see Theorem 6.3 therein), we here provide another (different and more elementary) proof for the reader convenience.

Theorem 4.1. *Let V_0 be in $\dot{B}_{2,1}^0(\mathbb{R}^2)$ with $\operatorname{div} V_0 = 0$. Then there exists a unique solution to (1.3) such that*

$$(4.69) \quad V \in L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^0(\mathbb{R}^2)) \quad \text{and} \quad V_t, \nabla^2 V \in L_1(\mathbb{R}_+; \dot{B}_{2,1}^0(\mathbb{R}^2)).$$

Furthermore, the following inequality is satisfied for all $T \geq 0$:

$$(4.70) \quad \|V\|_{L_\infty(0,T;\dot{B}_{2,1}^0)} + \|V_t, \nabla^2 V\|_{L_1(0,T;\dot{B}_{2,1}^0)} \leq C \|V_0\|_{\dot{B}_{2,1}^0} \exp\left(\frac{C}{\mu^4} \|V_0\|_{L_2}^4\right)$$

for some universal constant C .

Proof. First recall that the space $\dot{B}_{2,1}^0(\mathbb{R}^2)$ for initial velocity embeds in $L_2(\mathbb{R}^2)$. Hence we have $V_0 \in L_2(\mathbb{R}^2)$ and the pioneering works by J. Leray in [17] delivers us global in time weak solutions satisfying the energy estimate

$$(4.71) \quad \sup_{t \in \mathbb{R}_+} \|V(t)\|_{L_2(\mathbb{R}^2)}^2 + 2\mu \int_0^\infty \|\nabla V\|_{L_2(\mathbb{R}^2)}^2 dx = \|V_0\|_{L_2(\mathbb{R}^2)}^2.$$

Then the classical result of Olga Alexandrovna [16] provides a unique global in time solution $V \in L_\infty(\mathbb{R}_+; L_2(\mathbb{R}^2)) \cap L_2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^2))$. Here we want to improve the regularity to the class defined by (4.69).

Now, real interpolation applied to (4.71) gives

$$V \in \left(L_\infty(\mathbb{R}_+; L_2(\mathbb{R}^2)), L_2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^2)) \right)_{1/2,1},$$

which implies that $V \in L_4(\mathbb{R}_+; \dot{B}_{2,1}^{1/2}(\mathbb{R}^2))$.

To take advantage of that information, we look at the equation satisfied by V as a nonlinear modification of the Stokes system, namely

$$(4.72) \quad \begin{aligned} V_t - \mu \Delta V + \nabla \Pi &= -V \cdot \nabla V, \\ \operatorname{div} V &= 0, \\ V|_{t=0} &= V_0. \end{aligned}$$

Using the endpoint maximal regularity estimates of the Stokes system in homogeneous Besov spaces (which, in \mathbb{R}^2 , coincide with those for the heat equation), we may write

$$(4.73) \quad \|V\|_{L_\infty(0,T;\dot{B}_{2,1}^0)} + \|V_t, \mu \nabla^2 V\|_{L_1(0,T;\dot{B}_{2,1}^0)} \leq C (\|V \cdot \nabla V\|_{L_1(0,T;\dot{B}_{2,1}^0)} + \|V_0\|_{\dot{B}_{2,1}^0}).$$

Next we have to bound $\|V \cdot \nabla V\|_{L_1(0,T;\dot{B}_{2,1}^0)}$. From the energy balance (4.71) and the interpolation property pointed out above, we know that

$$(4.74) \quad \mu^{1/4} \|V\|_{L_4(\mathbb{R}_+;\dot{B}_{2,1}^{1/2})} \leq C \|V_0\|_{L_2}.$$

Furthermore, product laws in Besov spaces (Lemma 2.1) ensure that

$$(4.75) \quad \|V \cdot \nabla V\|_{\dot{B}_{2,1}^0} \leq C \|V\|_{\dot{B}_{2,1}^{1/2}} \|\nabla V\|_{\dot{B}_{2,1}^{1/2}}.$$

Hence integrating (4.75) on the time interval $[0, T]$, and using the following interpolation inequality:

$$\|Z\|_{\dot{B}_{2,1}^{1/2}} \leq C \|Z\|_{\dot{B}_{2,1}^{-1}}^{1/4} \|\nabla Z\|_{\dot{B}_{2,1}^0}^{3/4},$$

together with Young inequality, we may write for all $\varepsilon > 0$,

$$\begin{aligned} \|V \cdot \nabla V\|_{L_1(0,T;\dot{B}_{2,1}^0)} &\leq C \int_0^T \|V\|_{\dot{B}_{2,1}^{1/2}} \|\nabla V\|_{\dot{B}_{2,1}^{1/2}} dt \\ &\leq C \int_0^T \|V\|_{\dot{B}_{2,1}^{1/2}} \|\nabla V\|_{\dot{B}_{2,1}^{-1}}^{1/4} \|\nabla^2 V\|_{\dot{B}_{2,1}^0}^{3/4} dt \\ &\leq \frac{C}{\varepsilon^3 \mu^3} \int_0^T \|V\|_{\dot{B}_{2,1}^{1/2}}^4 \|V\|_{\dot{B}_{2,1}^0} dt + \varepsilon \mu \|\nabla^2 V\|_{L_1(0,T;\dot{B}_{2,1}^0)}. \end{aligned}$$

Hence, reverting to (4.73) and taking ε small enough, we find that

$$(4.76) \quad \|V\|_{L_\infty(0,T;\dot{B}_{2,1}^0)} + \|V_t, \mu \nabla^2 V\|_{L_1(0,T;\dot{B}_{2,1}^0)} \leq C \left(\frac{1}{\mu^3} \int_0^T \|V\|_{\dot{B}_{2,1}^{1/2}}^4 \|V\|_{\dot{B}_{2,1}^0} dt + \|V_0\|_{\dot{B}_{2,1}^0} \right).$$

In view of the Gronwall inequality, this gives

$$(4.77) \quad \|V\|_{L_\infty(0,T;\dot{B}_{2,1}^0(\mathbb{R}^2))} \leq C \|V_0\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \exp \left(\frac{C}{\mu^3} \int_0^T \|V\|_{\dot{B}_{2,1}^{1/2}}^4 dt \right).$$

Remembering (4.74), one can conclude to (4.70).

For the sake of completeness, we have to prove that $\nabla^2 V \in L_1(\mathbb{R}_+; \dot{B}_{2,1}^0)$ as it has been assumed implicitly in the above computations. This may be obtained by bootstrap from the property that $V \in L_4(\mathbb{R}_+; \dot{B}_{2,1}^{1/2})$. Indeed, because $V \cdot \nabla V = \operatorname{div}(V \otimes V)$, Lemma 2.1 gives $V \cdot \nabla V \in L_2(\mathbb{R}_+; \dot{B}_{2,1}^{-1})$, and thus $V \in L_2(\mathbb{R}_+; \dot{B}_{2,1}^1)$ through (4.72), thanks to maximal regularity results for the Stokes system. Knowing that $V \in L_2(\mathbb{R}_+; \dot{B}_{2,1}^1)$, Lemma 2.1 now gives us $V \cdot \nabla V \in L_1(\mathbb{R}_+; \dot{B}_{2,1}^0)$, and thus $(V_t, \nabla^2 V) \in L_1(\mathbb{R}_+; \dot{B}_{2,1}^0)$. \square

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